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Quantitative unique continuation for the semilinear heat equation in a convex domain [☆]

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Abstract

In this paper, we study certain unique continuation properties for solutions of the semilinear heat equation $\partial_t u - \Delta u = g(u)$, with the homogeneous Dirichlet boundary condition, over $\Omega \times (0, T_*)$. Ω is a bounded, convex open subset of \mathbb{R}^d , with a smooth boundary for the subset. The function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfies certain conditions. We establish some observation estimates for $(u - v)$, where u and v are two solutions to the above-mentioned equation. The observation is made over $\omega \times \{T\}$, where ω is any non-empty open subset of Ω , and T is a positive number such that both u and v exist on the interval $[0, T]$. At least two results can be derived from these estimates: (i) if $\|(u - v)(\cdot, T)\|_{L^2(\omega)} = \delta$, then $\|(u - v)(\cdot, T)\|_{L^2(\Omega)} \leq C\delta^\alpha$ where constants $C > 0$ and $\alpha \in (0, 1)$ can be independent of u and v in certain cases; (ii) if two solutions of the above equation hold the same value over $\omega \times \{T\}$, then they coincide over $\Omega \times [0, T_m]$. T_m indicates the maximum number such that these two solutions exist on $[0, T_m]$.

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1. Introduction

In this paper, we establish some quantitative unique continuation theorems for the semilinear heat equation. Let us start with introducing the equation. Let Ω be a bounded, convex open subset of \mathbb{R}^d , $d \geq 1$, with a smooth boundary $\partial\Omega$ for the subset. We consider the following semilinear heat equation:

$$\begin{cases} \partial_t u - \Delta u = g(u) & \text{in } \Omega \times (0, T_*), \\ u = 0 & \text{on } \partial\Omega \times (0, T_*). \end{cases} \quad (1.1)$$

Here, T_* is a positive number and the function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfies one of the following two conditions:

(H_1) The function g is of class $C^1(\mathbb{R})$. Furthermore, there are positive numbers c and p with $1 < p \leq 4/d + 1$ such that

$$|g(y_1) - g(y_2)| \leq c(1 + |y_1|^{p-1} + |y_2|^{p-1})|y_1 - y_2| \quad \text{for all } y_1, y_2 \in \mathbb{R}.$$

(H_2) The function g is locally Lipschitz.

With regards to Eq. (1.1), we recall the following results (see [1]):

(i) When g satisfies the condition (H_1), for each $u_0 \in L^2(\Omega)$, Eq. (1.1) has a unique solution u only when $u(\cdot, 0) = u_0(\cdot)$; u is defined on a maximal interval $[0, T_m(u_0))$; and for each $T \in (0, T_m(u_0))$, $u \in C([0, T]; L^2(\Omega)) \cap L_{loc}^\infty((0, T); L^\infty(\Omega))$ and it holds that

$$\|u(\cdot, t)\|_{L^2(\Omega)} + t^{d/4} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{(\Omega, T, \|u_0\|_{L^2(\Omega)}, p)} \quad \text{for each } t \in (0, T]. \quad (1.2)$$

(ii) When g satisfies the condition (H_2), for each u_0 in $L^\infty(\Omega)$, Eq. (1.1) has a unique solution u only when $u(\cdot, 0) = u_0(\cdot)$; u is defined on a maximal time interval $[0, T_m(u_0))$; and for each $T \in (0, T_m(u_0))$, $u \in L^\infty(\Omega \times (0, T))$ with the estimate:

$$\|u\|_{L^\infty(\Omega \times (0, T))} \leq C_{(\Omega, T, \|u_0\|_{L^\infty(\Omega)}, g)}. \quad (1.3)$$

Throughout this paper, $C_{(\dots)}$ stands for a positive constant that depends on what are enclosed in the brackets. For each solution u , with $u(\cdot, 0) = u_0(\cdot)$, to Eq. (1.1), we denote its maximal interval of the existence by $[0, T_m(u_0))$. When u and v are two solutions, with the maximal intervals of the existence $[0, T_m(u_0))$ and $[0, T_m(v_0))$ respectively, to Eq. (1.1), we write $T_m = \min\{T_m(u_0), T_m(v_0)\}$. The main results obtained in this study will be stated as two theorems. The first one is:

Theorem 1.1. *If g satisfies the condition (H_1), then any two solutions u and v , with both $u(\cdot, 0) \equiv u_0(\cdot)$ and $v(\cdot, 0) \equiv v_0(\cdot)$ belonging to $L^2(\Omega)$, of Eq. (1.1) have the property: for each T with $0 < T < T_m$, and for each non-empty open subset ω of Ω ,*

$$\int_{\Omega} |(u - v)(x, T)|^2 dx \leq C \left(\int_{\omega} |(u - v)(x, T)|^2 dx \right)^\alpha.$$

Here, $\alpha = \alpha_{(\Omega, \omega)} \in (0, 1)$ and $C = C_{(\Omega, \omega, T, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{L^2(\Omega)}, p)}$. Additionally, if $u(\cdot, T) = v(\cdot, T)$ over ω , then $u = v$ over $\Omega \times [0, T_m]$.

The above theorem provides the following valuable information about Eq. (1.1): if one can measure the difference between two solutions u and v , with $u(\cdot, 0)$ and $v(\cdot, 0)$ belonging to $L^2(\Omega)$, to Eq. (1.1), over $\omega \times \{T\}$, namely, if one can measure $\|u(\cdot, T) - v(\cdot, T)\|_{L^2(\omega)}^2 = \delta$, then $\|u(\cdot, T) - v(\cdot, T)\|_{L^2(\Omega)}^2 \leq C\delta^\alpha$ can be derived and provides certain information on $(u - v)$ over $\Omega \times \{T\}$. It should be more important in applications when the constant C is independent of the $u(\cdot, 0)$ and $v(\cdot, 0)$. We will see in the next section (see Proposition 2.2) that it is true for a class of solutions to Eq. (1.1) where $g(u) = |u|^{p-1}u$, and where certain suitable assumptions on p and Ω are made. Indeed, in that case, we can have $C = ce^{c/T} / T^{2(1-\alpha)/(p-1)}$ with $c = c_{(\Omega, \omega, p)}$.

The other theorem concerns a quantitative unique continuation estimate for the nonlinear system (1.1).

Theorem 1.2. *If g satisfies the condition (H_2) , then any two solutions u and v , with both $u(\cdot, 0) \equiv u_0(\cdot)$ and $v(\cdot, 0) \equiv v_0(\cdot)$ belonging to $L^\infty(\Omega)$ and $u_0 \neq v_0$, to Eq. (1.1) hold the property: for each T with $0 < T < T_m$, and for each non-empty open subset ω of Ω ,*

$$\|u_0 - v_0\|_{L^2(\Omega)}^2 \leq C \exp\left(C \frac{\|u_0 - v_0\|_{L^2(\Omega)}^2}{\|u_0 - v_0\|_{H^{-1}(\Omega)}^2}\right) \int_\omega |(u - v)(x, T)|^2 dx.$$

Here, $\alpha = \alpha_{(\Omega, \omega)} \in (0, 1)$ and $C = C_{(\Omega, \omega, T, \|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}, g)}$.

Clearly, Theorem 1.2 indicates that if two solutions, with initial values in $L^\infty(\Omega)$, to Eq. (1.1) where g is a locally Lipschitz function, hold the same value over $\omega \times \{T\}$, then they coincide over $\Omega \times [0, T_m]$. The unique continuation property in Theorem 1.2 is different from those introduced in [3,4].

The proof of Theorem 1.1 and Theorem 1.2 is based on certain quantitative unique continuation estimates for the linear heat equation with a potential $a(x, t)$. The estimates are established in Proposition 2.1 in the next section. We would like to conclude that if $g(u) = au$ with $a \in L^\infty(\Omega \times (0, T))$, then the constant C in Theorem 1.1 has the form

$$ce^{(c/T + c(T\|a\|_{L^\infty(\Omega \times (0, T))} + T^2\|a\|_{L^\infty(\Omega \times (0, T))}^2))} \|u_0 - v_0\|_{L^2(\Omega)}^{2(1-\alpha)}$$

with $c = c_{(\Omega, \omega)}$ (see the proof of Theorem 1.1 and Proposition 2.1).

The rest of the paper is constructed as: In Section 2, we first introduce quantitative unique continuation estimates for the linear heat equation with a potential, then prove Theorem 1.1 and Theorem 1.2 followed by some applications of Theorem 1.1. Section 3 presents the final proof of the above-mentioned estimates for the linear case.

2. Proof of the main results

We start with introducing certain unique continuation estimates for the heat equation with a potential. Let Ω be the domain introduced in Section 1. We recall that Ω is convex. Let L be a positive number and a be a function in the space $L^\infty(\Omega \times (0, L))$. Consider the linear equation:

$$\begin{cases} \partial_t \varphi - \Delta \varphi - a\varphi = 0 & \text{in } \Omega \times (0, L), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, L). \end{cases} \quad (2.1)$$

With regard to Eq. (2.1), we have:

Proposition 2.1. *Let ω be a non-empty open subset of Ω . Then there are two positive numbers $\alpha = \alpha_{(\Omega, \omega)} \in (0, 1)$ and $C = C_{(\Omega, \omega)}$ such that for each $L > 0$ and for each potential $a \in L^\infty(\Omega \times (0, L))$, any solution φ with $\varphi(\cdot, 0) \equiv \varphi_0(\cdot) \in L^2(\Omega)$, to Eq. (2.1) has the following estimate:*

$$\begin{aligned} \int_{\Omega} |\varphi(x, L)|^2 dx &\leq C \exp(C/L + C(L\|a\|_{L^\infty(\Omega \times (0, L))} + L^2\|a\|_{L^\infty(\Omega \times (0, L))}^2)) \\ &\quad \times \left(\int_{\Omega} |\varphi_0(x)|^2 dx \right)^{1-\alpha} \left(\int_{\omega} |\varphi(x, L)|^2 dx \right)^{\alpha}. \end{aligned}$$

Furthermore, if $\varphi_0 \neq 0$, then it holds that

$$\|\varphi_0\|_{L^2(\Omega)}^2 \leq C e^{C/L} \exp\left(C(L + \sqrt{L})e^{CL\|a\|_{L^\infty(\Omega \times (0, L))}} \left(\frac{\|\varphi_0\|_{L^2(\Omega)}^2}{\|\varphi_0\|_{H^{-1}(\Omega)}^2}\right)\right) \int_{\omega} |\varphi(x, L)|^2 dx.$$

We shall leave the proof of Proposition 2.1 till later (see Section 3). Now we turn to prove Theorem 1.1.

Proof of Theorem 1.1. Let u and v be two solutions to Eq. (1.1) such that both $u(\cdot, 0) \equiv u_0(\cdot)$ and $v(\cdot, 0) \equiv v_0(\cdot)$ are in $L^2(\Omega)$. We arbitrary fix a time T with $0 < T < T_m$. Define a function $a : \Omega \times (0, T) \rightarrow \mathbb{R}$ by setting

$$a(x, t) = \begin{cases} \frac{g(u(x, t)) - g(v(x, t))}{u(x, t) - v(x, t)} & \text{if } u(x, t) \neq v(x, t) \text{ and } (x, t) \in \Omega \times (0, T), \\ g'(u(x, t)) & \text{if } u(x, t) = v(x, t) \text{ and } (x, t) \in \Omega \times (0, T). \end{cases} \quad (2.2)$$

Clearly, the function a is measurable over $\Omega \times (0, T)$. By the condition (H_1) , we have

$$|g'(y)| \leq c(1 + 2|y|^{p-1}) \quad \text{for all } y \in \mathbb{R},$$

which, together with (2.2), yields the estimate:

$$|a(x, t)| \leq c(1 + |u(x, t)|^{p-1} + |v(x, t)|^{p-1}) \quad \text{for all } (x, t) \in \Omega \times (0, T).$$

Then, we derive from (1.2) that for each $\varepsilon \in (0, T)$,

$$\|a\|_{L^\infty(\Omega \times (\varepsilon, T))} \leq C_{(\Omega, \varepsilon, T, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{L^2(\Omega)}, p)}. \quad (2.3)$$

Let $\Phi = u - v$. For each ε given in $(0, T/2]$, we define a function φ_ε on $\Omega \times [0, T - \varepsilon]$ by setting $\varphi_\varepsilon(x, t) = \Phi(x, t + \varepsilon)$. It is obvious that φ_ε satisfies

$$\begin{cases} \partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon - a(x, t + \varepsilon) \varphi_\varepsilon = 0 & \text{in } \Omega \times (0, T - \varepsilon), \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T - \varepsilon), \\ \varphi_\varepsilon(\cdot, 0) = \Phi(\cdot, \varepsilon) & \text{in } \Omega. \end{cases}$$

Then, an application of Proposition 2.1 to the above equation leads to the following two estimates:

$$\begin{aligned} \int_{\Omega} |\Phi(x, T)|^2 dx &\leq C \exp(C/T + C(T\|a\|_{L^\infty(\Omega \times (\varepsilon, T))} + T^2\|a\|_{L^\infty(\Omega \times (\varepsilon, T))}^2)) \\ &\quad \times \left(\int_{\Omega} |\Phi(x, \varepsilon)|^2 dx \right)^{1-\alpha} \left(\int_{\omega} |\Phi(x, T)|^2 dx \right)^{\alpha} \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \|\Phi(\cdot, \varepsilon)\|_{L^2(\Omega)}^2 &\leq C e^{C/T} \exp\left(C(T + \sqrt{T}) e^{CT\|a\|_{L^\infty(\Omega \times (\varepsilon, T))}^2} \left(\frac{\|\Phi(\cdot, \varepsilon)\|_{L^2(\Omega)}^2}{\|\Phi(\cdot, \varepsilon)\|_{H^{-1}(\Omega)}^2} \right)\right) \\ &\quad \times \int_{\omega} |\Phi(x, T)|^2 dx \end{aligned} \quad (2.5)$$

when $\Phi(\cdot, \varepsilon) \neq 0$.

Next, we utilize (1.2) to get an estimate of $\|\Phi(\cdot, T/2)\|_{L^2(\Omega)}$ in terms of T , $\|u_0\|_{L^2(\Omega)}$ and $\|v_0\|_{L^2(\Omega)}$. Combining with this estimate and (2.3), the inequality (2.4) with $\varepsilon = T/2$, gives the desired estimate of Theorem 1.1.

Finally, we shall prove the unique continuation. From (2.3) and (2.5), we indicate that if $u(x, T) = v(x, T)$ for a.e. x in ω , then $u(x, \varepsilon) = v(x, \varepsilon)$ for a.e. x in Ω . Since both u and v belong to $C([0, T]; L^2(\Omega))$, we necessarily have $0 = \lim_{\varepsilon \rightarrow 0} \|u(\cdot, \varepsilon) - v(\cdot, \varepsilon)\|_{L^2(\Omega)} = \|u(\cdot, 0) - v(\cdot, 0)\|_{L^2(\Omega)}$. Hence, $u_0 = v_0$. This completes the proof. \square

Remark. With the proof of Theorem 1.1 (in particular, the inequality (2.4)), we can expect a more precise estimate in Theorem 1.1 if more information about the bounds of the terms $\|a\|_{L^\infty(\Omega \times (T/2, T))}$, $\|u(\cdot, T/2)\|_{L^2(\Omega)}$, and $\|v(\cdot, T/2)\|_{L^2(\Omega)}$ is available. This expectation can be met provided that one holds an explicit expression for the bound of $|u(x, t)|$, $(x, t) \in \Omega \times [T/2, T]$, in terms of T and/or u_0 . Then, we would like to give a refined estimate corresponding to that of Theorem 1.1 in certain cases. We first recall some existing results for the solutions u , with $u(\cdot, 0) \equiv u_0(\cdot) \in L^\infty(\Omega)$, to Eq. (1.1) where $g(u) = |u|^{p-1}u$, $p > 1$:

- (i) u is a classical solution, namely, $u \in C^{2,1}(\overline{\Omega} \times (0, T))$.
- (ii) If $u_0 \geq 0$, then u is a nonnegative solution (i.e., $u \geq 0$).
- (iii) In certain cases, this equation can have a global solution u , namely, $T_m(u_0) = \infty$ (see e.g., [8]).
- (iv) Polacik, Quittner and Souplet proved in [6] that if $(d-1)^2 p < d(d+2)$, then any nonnegative global classical solution u to this equation has the estimate:

$$|u(x, t)| \leq C_{(\Omega, p)} \left(1 + \frac{1}{t^{\frac{1}{p-1}}} \right) \quad \forall (x, t) \in \Omega \times (0, +\infty).$$

Moreover, this estimate holds also for the case that u is further radial, $(d-2)p < d+2$ and Ω is symmetric.

Then, we can easily obtain the following result based on (iv) and (2.4).

Proposition 2.2.

- (i) Let p be a number such that $1 < p$ and $(d-1)^2 p < d(d+2)$. Then, any two nonnegative global classical solutions u and v of Eq. (1.1) where $g(u) = u^p$, have the estimate:

$$\int_{\Omega} |(u-v)(x, T)|^2 dx \leq C \frac{1}{T^{2(1-\alpha)/(p-1)}} e^{C/T} \left(\int_{\omega} |(u-v)(x, T)|^2 dx \right)^{\alpha}.$$

Here, T is an arbitrary positive number, $\alpha = \alpha_{(\Omega, \omega)} \in (0, 1)$ and $C = C_{(\Omega, \omega, p)}$.

- (ii) Let p be a number such that $1 < p$ and $(d-2)p < d+2$. Suppose that Ω is symmetric. Then, any two nonnegative global radial classical solutions u and v of Eq. (1.1) where $g(u) = u^p$, also hold the above-mentioned estimate.

Now we turn to prove Theorem 1.2.

Proof of Theorem 1.2. Let u and v be two solutions to Eq. (1.1) such that both $u(\cdot, 0) \equiv u_0(\cdot)$ and $v(\cdot, 0) \equiv v_0(\cdot)$ are in $L^{\infty}(\Omega)$. Let T be a positive number such that $T < T_m$. Then, both u and v belong to the space $L^{\infty}(\Omega \times (0, T))$. Furthermore, it follows from (1.3) that

$$\|u\|_{L^{\infty}(\Omega \times (0, T))} + \|v\|_{L^{\infty}(\Omega \times (0, T))} \leq C_{(\Omega, T, \|u_0\|_{L^{\infty}(\Omega)}, \|v_0\|_{L^{\infty}(\Omega)}, g)} \equiv M.$$

Because of (H_2) , corresponding to M , there exists a constant $E \equiv E_{(\Omega, T, \|u_0\|_{L^{\infty}(\Omega)}, \|v_0\|_{L^{\infty}(\Omega)}, g)}$ such that

$$|g(y_1) - g(y_2)| \leq E|y_1 - y_2| \quad \text{for all } y_1, y_2 \text{ in } [-M, M]. \quad (2.6)$$

Write

$$a(x, t) = \begin{cases} \frac{g(u(x, t)) - g(v(x, t))}{u(x, t) - v(x, t)} & \text{if } u(x, t) \neq v(x, t) \text{ and } (x, t) \in \Omega \times (0, T), \\ 0 & \text{if } u(x, t) = v(x, t) \text{ and } (x, t) \in \Omega \times (0, T). \end{cases} \quad (2.7)$$

Clearly, it follows from (2.6) and (2.7) that $a \in L^{\infty}(\Omega \times (0, T))$ with $\|a\|_{L^{\infty}(\Omega \times (0, T))} \leq E$. Furthermore, it holds that $g(u) - g(v) = a(u - v)$ over $\Omega \times (0, T)$. Set $\varphi = u - v$. Then φ solves (2.1) and satisfies $\varphi(\cdot, 0) = u_0(\cdot) - v_0(\cdot)$. Now an application of Proposition 2.1 to the solution φ gives the desired estimate. This completes the proof. \square

3. Proof of Proposition 2.1

The main idea to prove Proposition 2.1 originates from the papers [7,2]. We begin with introducing a technical lemma, which is the base of the proof to Proposition 2.1. For this purpose, we fix a positive number r and a point x_0 in the subset ω such that $B_r \subset \omega$. B_r denotes the open ball, centered at the point x_0 and of radius r , in \mathbb{R}^d . Write $m = \sup_{x \in \Omega} |x - x_0|^2$.

Lemma 3.1. For each $L > 0$ and $\varphi_0 \in L^2(\Omega)$, the solution φ , with $\varphi(\cdot, 0) = \varphi_0(\cdot)$, to Eq. (2.1) holds the estimate:

$$\begin{aligned} & \left[1 - \frac{8\lambda}{r^2} \left(\frac{\lambda}{L} + 1 \right) \mathcal{K}_{a,\varphi,L} \right] \int_{\Omega} |x - x_0|^2 |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\ & \leq 8\lambda \left(\frac{\lambda}{L} + 1 \right) \mathcal{K}_{a,\varphi,L} \int_{B_r} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx, \end{aligned}$$

where λ is an arbitrary positive number and

$$\mathcal{K}_{a,\varphi,L} \equiv 2 \ln \left(\frac{\int_{\Omega} |\varphi(x, 0)|^2 dx}{\int_{\Omega} |\varphi(x, L)|^2 dx} \right) + \mathcal{D}_{a,L} + \frac{m}{L} + \frac{d}{2},$$

with

$$\mathcal{D}_{a,L} \equiv 4L \|a\|_{L^\infty(\Omega \times (0,L))} + L^2 \|a\|_{L^\infty(\Omega \times (0,L))}^2.$$

We shall leave the proof of Lemma 3.1 till later. Now we are on the position to prove Proposition 2.1.

Proof of Proposition 2.1. We start with proving the first estimate in the proposition. By taking $\lambda > 0$ in the estimate of Lemma 3.1 to be such that

$$\frac{8\lambda}{r^2} \left(\frac{\lambda}{L} + 1 \right) \mathcal{K}_{a,\varphi,L} = \frac{1}{2}, \quad (3.1)$$

we get

$$\int_{\Omega} |x - x_0|^2 |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \leq r^2 \int_{B_r} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx.$$

Then, we derive that

$$\begin{aligned} \int_{\Omega} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx & \leq \int_{\Omega \cap \{|x-x_0| \geq r\}} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx + \int_{B_r} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\ & \leq \frac{1}{r^2} \int_{\Omega} |x - x_0|^2 |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx + \int_{B_r} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\ & \leq 2 \int_{B_r} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx. \end{aligned}$$

Consequently, we obtain the following inequality:

$$\int_{\Omega} |\varphi(x, L)|^2 dx \leq 2e^{\frac{m}{4\lambda}} \int_{B_r} |\varphi(x, L)|^2 dx. \quad (3.2)$$

It remains to estimate the term $e^{\frac{m}{4\lambda}}$. To this end, we solve Eq. (3.1) for $\lambda > 0$ to get

$$\lambda = \frac{1}{2} \left(-L + \sqrt{L^2 + \frac{Lr^2}{4\mathcal{K}_{a,\varphi,L}}} \right).$$

Since $\frac{m}{L} \leq \mathcal{K}_{a,\varphi,L}$, it follows that

$$\begin{aligned} \frac{1}{\lambda} &= 2 \frac{L + \sqrt{L^2 + \frac{Lr^2}{4\mathcal{K}_{a,\varphi,L}}}}{\frac{Lr^2}{4\mathcal{K}_{a,\varphi,L}}} \leq 8 \left(2L + \sqrt{\frac{Lr^2}{4\mathcal{K}_{a,\varphi,L}}} \right) \frac{1}{Lr^2} \mathcal{K}_{a,\varphi,L} \\ &\leq \left(16 + \frac{4r}{\sqrt{m}} \right) \frac{1}{r^2} \mathcal{K}_{a,\varphi,L}. \end{aligned}$$

Therefore, it holds that

$$\begin{aligned} e^{\frac{m}{4\lambda}} &\leq e^{(4m+r\sqrt{m}) \frac{1}{r^2} \mathcal{K}_{a,\varphi,L}} \\ &\leq e^{(4m+r\sqrt{m}) \frac{1}{r^2} \frac{d}{2}} e^{(4m+r\sqrt{m}) \frac{1}{r^2} (\frac{m}{L} + \mathcal{D}_{a,L})} \left(\frac{\int_{\Omega} |\varphi(x, 0)|^2 dx}{\int_{\Omega} |\varphi(x, L)|^2 dx} \right)^{2(4m+r\sqrt{m})/r^2}. \end{aligned} \quad (3.3)$$

Consequently, from (3.2) and (3.3), we get the estimate:

$$\begin{aligned} \int_{\Omega} |\varphi(x, L)|^2 dx &\leq 2e^{(4m+r\sqrt{m}) \frac{1}{r^2} \frac{d}{2}} e^{(4m+r\sqrt{m}) \frac{1}{r^2} (\frac{m}{L} + \mathcal{D}_{a,L})} \\ &\quad \times \left(\frac{\int_{\Omega} |\varphi(x, 0)|^2 dx}{\int_{\Omega} |\varphi(x, L)|^2 dx} \right)^{2(4m+r\sqrt{m})/r^2} \int_{B_r} |\varphi(x, L)|^2 dx. \end{aligned} \quad (3.4)$$

Then by (3.4) and by the definition of $\mathcal{D}_{a,L}$, we have

$$\begin{aligned} \int_{\Omega} |\varphi(x, L)|^2 dx &\leq C e^{\frac{C}{r^2}} e^{\frac{C}{Lr^2}} e^{\frac{C}{r^2} (L\|a\|_{L^\infty(\Omega \times (0,L))} + L^2\|a\|_{L^\infty(\Omega \times (0,L))}^2)} \\ &\quad \times \left(\frac{\int_{\Omega} |\varphi(x, 0)|^2 dx}{\int_{\Omega} |\varphi(x, L)|^2 dx} \right)^{C/r^2} \int_{B_r} |\varphi(x, L)|^2 dx \end{aligned}$$

which is equivalent to the following inequality:

$$\int_{\Omega} |\varphi(x, L)|^2 dx \leq C e^{C/L} e^{C(L\|a\|_{L^\infty(\Omega \times (0, L))} + L^2\|a\|_{L^\infty(\Omega \times (0, L))}^2)} \\ \times \left(\int_{\Omega} |\varphi(x, 0)|^2 dx \right)^{\frac{C}{r^2+C}} \left(\int_{B_r} |\varphi(x, L)|^2 dx \right)^{\frac{r^2}{r^2+C}}. \quad (3.5)$$

Because $B_r \subset \omega$, the above inequality gives the first estimate of the proposition. \square

Next, we turn to prove the second estimate in the proposition. Recall the following backward uniqueness estimate: (It is a direct consequence of the estimate (3.12) in Ref. [5].)

$$\frac{\int_{\Omega} |\varphi(x, 0)|^2 dx}{\int_{\Omega} |\varphi(x, L)|^2 dx} \leq \frac{1}{L} \exp \left(c e^{cL\|a\|_{L^\infty(\Omega \times (0, T))}^2} (L + \sqrt{L}) \left(\frac{\|\varphi_0\|_{L^2(\Omega)}^2}{\|\varphi_0\|_{H^{-1}(\Omega)}^2} \right) \right), \quad (3.6)$$

where $c = c_{(\Omega)} > 1$. Hence, (3.6), together with (3.5), yields the second estimate of the proposition, and completes the proof of the proposition.

Finally, we prove Lemma 3.1.

Proof of Lemma 3.1. We first point out the following fact: for any $f \in H_0^1(\Omega)$ and for each $\lambda > 0$, it holds that $0 \leq \int_{\Omega} |\nabla(f(x) \exp(-\frac{|x-x_0|^2}{8\lambda}))|^2 dx$. By computing the right-hand term, we get

$$\int_{\Omega} \frac{|x-x_0|^2}{8\lambda} |f(x)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \leq 2\lambda \int_{\Omega} |\nabla f(x)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\ + \frac{d}{2} \int_{\Omega} |f(x)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx. \quad (3.7)$$

Now, we arbitrary fix a positive number L and a function φ_0 in $L^2(\Omega)$ such that $\varphi_0 \neq 0$. Let $\varphi = \varphi(x, t)$ be the solution of Eq. (2.1) such that $\varphi(\cdot, 0) = \varphi_0(\cdot)$ over Ω . Corresponding to each $\lambda > 0$, we first introduce the following weight function over $\mathbb{R}^d \times [0, L]$:

$$G_\lambda(x, t) = \frac{1}{(L-t+\lambda)^{d/2}} e^{-\frac{|x-x_0|^2}{4(L-t+\lambda)}}, \quad (3.8)$$

and then, define the following three functions over the interval $[0, L]$:

$$H_\lambda(t) = \int_{\Omega} |\varphi(x, t)|^2 G_\lambda(x, t) dx, \quad (3.9)$$

$$D_\lambda(t) = \int_{\Omega} |\nabla \varphi(x, t)|^2 G_\lambda(x, t) dx, \quad (3.10)$$

and

$$N_\lambda(t) = \frac{2D_\lambda(t)}{H_\lambda(t)}. \quad (3.11)$$

It follows from (3.7) and (3.11) that

$$\begin{aligned} & \int_{\Omega} |x - x_0|^2 |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\ & \leq 8\lambda \left(2\lambda \int_{\Omega} |\nabla \varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx + \frac{d}{2} \int_{\Omega} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \right) \\ & \leq 8\lambda \left(\lambda N_\lambda(L) + \frac{d}{2} \right) \int_{\Omega} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \\ & \leq 8\lambda \left(\lambda N_\lambda(L) + \frac{d}{2} \right) \left[\int_{B_r} |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \right. \\ & \quad \left. + \frac{1}{r^2} \int_{\Omega \setminus B_r} |x - x_0|^2 |\varphi(x, L)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \right]. \end{aligned} \quad (3.12)$$

Next, we are going to show the following estimate:

$$\lambda N_\lambda(L) + \frac{d}{2} \leq \left(\frac{\lambda}{L} + 1 \right) \mathcal{K}_{a,\varphi,L}. \quad (3.13)$$

If (3.13) holds, then the estimate of Lemma 3.1 follows immediately from (3.12) and (3.13), and we complete the proof of Lemma 3.1.

The proof of the estimate (3.13) will be carried out by seven steps. In Step 1, we shall give certain identities related to the weight function G_λ . In Step 2, we prove an identity for the term $\frac{d}{dt} \ln H_\lambda$. In Step 3 and Step 4, we compute terms $\frac{d}{dt} H_\lambda D_\lambda$ and $\frac{d}{dt} D_\lambda$ respectively. In Step 5, we will deduce the derivative of N_λ . In Step 6, we shall give a bound for the term $\frac{d}{dt} [(L - t + \lambda)N_\lambda(t)]$. In Step 7, we derive the desired estimate for the term $(\lambda N_\lambda(L) + \frac{d}{2})$. Now, we start with the first step.

Step 1. Properties of the function G_λ .

By direct computations, one can easily check that for each $\lambda > 0$, the function G_λ given by (3.8) holds the following four identities over $\mathbb{R}^d \times [0, L]$:

$$\partial_t G_\lambda(x, t) + \Delta G_\lambda(x, t) = 0, \quad (3.14)$$

$$\nabla G_\lambda(x, t) = \frac{-(x - x_0)}{2(L - t + \lambda)} G_\lambda(x, t), \quad (3.15)$$

$$\partial_t^2 G_\lambda(x, t) = \frac{-1}{2(L - t + \lambda)} G_\lambda(x, t) + \frac{|x_i - x_{0i}|^2}{4(L - t + \lambda)^2} G_\lambda(x, t), \quad (3.16)$$

and for $i \neq j$,

$$\partial_i \partial_j G_\lambda(x, t) = \frac{(x_i - x_{0i})(x_j - x_{0j})}{4(L - t + \lambda)^2} G_\lambda(x, t). \quad (3.17)$$

Step 2. Computation of $\frac{d}{dt} \ln H_\lambda(t)$ over the interval $[0, L]$.

By (3.9), (3.14) and the fact that $\varphi = 0$ on $\partial\Omega$, we get

$$\begin{aligned} H'_\lambda(t) &= 2 \int_{\Omega} \varphi \partial_t \varphi G_\lambda dx + \int_{\Omega} |\varphi|^2 \partial_t G_\lambda dx \\ &= 2 \int_{\Omega} \varphi \partial_t \varphi G_\lambda dx - \int_{\Omega} |\varphi|^2 \Delta G_\lambda dx \\ &= 2 \int_{\Omega} \varphi \partial_t \varphi G_\lambda dx - \int_{\Omega} \Delta |\varphi|^2 G_\lambda dx \\ &= 2 \int_{\Omega} \varphi (\partial_t \varphi - \Delta \varphi) G_\lambda dx - 2 \int_{\Omega} |\nabla \varphi|^2 G_\lambda dx. \end{aligned}$$

Therefore, for each $t \in [0, L]$, it holds that

$$\frac{d}{dt} \ln H_\lambda(t) = \frac{H'_\lambda(t)}{H_\lambda(t)} = -N_\lambda(t) + \frac{2}{H_\lambda(t)} \int_{\Omega} \varphi (\partial_t \varphi - \Delta \varphi) G_\lambda dx. \quad (3.18)$$

Step 3. Computation of $H'_\lambda(t) D_\lambda(t)$ over the interval $[0, L]$.

We aim to express both $H'_\lambda(t)$ and $D_\lambda(t)$ in term of $\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2}(\Delta \varphi - \partial_t \varphi)$. On one hand, by (3.9), (3.14), (3.15) and the fact that $\varphi = 0$ on $\partial\Omega$, we get another expression for the term $H'_\lambda(t)$ over $[0, L]$ as the following:

$$\begin{aligned} H'_\lambda(t) &= 2 \int_{\Omega} \varphi \partial_t \varphi G_\lambda dx - \int_{\Omega} |\varphi|^2 \Delta G_\lambda dx \\ &= 2 \int_{\Omega} \varphi \partial_t \varphi G_\lambda dx + \int_{\Omega} \nabla |\varphi|^2 \nabla G_\lambda dx \\ &= 2 \int_{\Omega} \varphi \left(\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi \right) G_\lambda dx \\ &= 2 \int_{\Omega} \varphi \left(\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2}(\Delta \varphi - \partial_t \varphi) \right) G_\lambda dx - \int_{\Omega} \varphi (\Delta \varphi - \partial_t \varphi) G_\lambda dx. \end{aligned}$$

On the other hand, from the definition of $D_\lambda(t)$ (see (3.10)), (3.15) and the fact that $\varphi = 0$ on $\partial\Omega$, we indicate that for each $t \in [0, L]$,

$$\begin{aligned} D_\lambda(t) &= \int_{\Omega} \nabla \varphi \nabla \varphi G_\lambda \, dx \\ &= \int_{\Omega} \operatorname{div}(\varphi \nabla \varphi G_\lambda) \, dx - \int_{\Omega} \varphi \operatorname{div}(\nabla \varphi G_\lambda) \, dx \\ &= - \int_{\Omega} \varphi \Delta \varphi G_\lambda \, dx + \int_{\Omega} \varphi \nabla \varphi \frac{x - x_0}{2(L - t + \lambda)} G_\lambda \, dx \\ &= - \int_{\Omega} \varphi \left(\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right) G_\lambda \, dx \\ &\quad - \frac{1}{2} \int_{\Omega} \varphi (\Delta \varphi - \partial_t \varphi) G_\lambda \, dx. \end{aligned}$$

Now, the aforementioned two expressions for terms $H'_\lambda(t)$ and $D_\lambda(t)$ yield that the following identity holds over $[0, L]$:

$$\begin{aligned} 2H'_\lambda(t)D_\lambda(t) &= - \left[2 \int_{\Omega} \varphi \left(\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right) G_\lambda \, dx \right]^2 \\ &\quad + \left[\int_{\Omega} \varphi (\Delta \varphi - \partial_t \varphi) G_\lambda \, dx \right]^2. \end{aligned} \quad (3.19)$$

Step 4. Computation of the term $D'_\lambda(t)$ over the interval $[0, L]$.

From the definition of the function $D_\lambda(t)$ (see (3.10)), and from (3.14), (3.15) and the fact that $\varphi = 0$ on $\partial\Omega$, we derive that

$$\begin{aligned} D'_\lambda(t) &= 2 \int_{\Omega} \nabla \varphi \partial_t \nabla \varphi G_\lambda \, dx + \int_{\Omega} |\nabla \varphi|^2 \partial_t G_\lambda \, dx \\ &= 2 \int_{\Omega} \nabla \varphi \partial_t \nabla \varphi G_\lambda \, dx - \int_{\Omega} |\nabla \varphi|^2 \Delta G_\lambda \, dx \\ &= 2 \int_{\Omega} \operatorname{div}(\partial_t \varphi \nabla \varphi G_\lambda) \, dx - 2 \int_{\Omega} \partial_t \varphi \operatorname{div}(\nabla \varphi G_\lambda) \, dx - \int_{\Omega} |\nabla \varphi|^2 \Delta G_\lambda \, dx \\ &= -2 \int_{\Omega} \partial_t \varphi \Delta \varphi G_\lambda \, dx - 2 \int_{\Omega} \partial_t \varphi \nabla \varphi \nabla G_\lambda \, dx - \int_{\Omega} |\nabla \varphi|^2 \Delta G_\lambda \, dx \end{aligned}$$

$$\begin{aligned}
&= -2 \int_{\Omega} \partial_t \varphi \Delta \varphi G_{\lambda} dx + 2 \int_{\Omega} \partial_t \varphi \nabla \varphi \frac{x - x_0}{2(L - t + \lambda)} G_{\lambda} dx \\
&\quad - \int_{\Omega} |\nabla \varphi|^2 \Delta G_{\lambda} dx.
\end{aligned} \tag{3.20}$$

Next, we turn to deal with the last term on the right-hand side of (3.20). For this purpose, we first observe the following pointwise identity:

$$\begin{aligned}
|\nabla \varphi|^2 \Delta G_{\lambda} &= \operatorname{div}(|\nabla \varphi|^2 \nabla G_{\lambda}) - 2 \operatorname{div}(\nabla \varphi (\nabla \varphi \nabla G_{\lambda})) \\
&\quad + 2 \Delta \varphi \nabla \varphi \nabla G_{\lambda} + 2 \sum_{i=1}^d \nabla \varphi \partial_i \varphi \partial_i \nabla G_{\lambda}.
\end{aligned} \tag{3.21}$$

Now, we write

$$\mathcal{B} \equiv \int_{\partial \Omega} |\nabla \varphi|^2 \partial_{\nu} G_{\lambda} d\sigma - 2 \int_{\partial \Omega} \partial_{\nu} \varphi (\nabla \varphi \nabla G_{\lambda}) d\sigma. \tag{3.22}$$

Then, by (3.21), (3.22), (3.15), (3.16) and (3.17), we obtain that

$$\begin{aligned}
\int_{\Omega} |\nabla \varphi|^2 \Delta G_{\lambda} dx &= \mathcal{B} + 2 \int_{\Omega} \Delta \varphi \nabla \varphi \nabla G_{\lambda} dx + 2 \sum_{i=1}^d \int_{\Omega} \nabla \varphi \partial_i \varphi \partial_i \nabla G_{\lambda} dx \\
&= \mathcal{B} - 2 \int_{\Omega} \Delta \varphi \nabla \varphi \frac{x - x_0}{2(L - t + \lambda)} G_{\lambda} dx \\
&\quad + 2 \sum_{i=1}^n \int_{\Omega} \partial_i \varphi \partial_i \varphi \left(\frac{-1}{2(L - t + \lambda)} G_{\lambda} + \frac{|x - x_0|^2}{4(L - t + \lambda)^2} G_{\lambda} \right) dx \\
&\quad + 2 \sum_{i \neq j} \int_{\Omega} \partial_j \varphi \partial_i \varphi \frac{(x_i - x_{0i})(x_j - x_{0j})}{4(L - t + \lambda)^2} G_{\lambda} dx \\
&= \mathcal{B} - 2 \int_{\Omega} \Delta \varphi \nabla \varphi \frac{x - x_0}{2(L - t + \lambda)} G_{\lambda} dx + \int_{\Omega} |\nabla \varphi|^2 \left(\frac{-1}{(L - t + \lambda)} \right) G_{\lambda} dx \\
&\quad + 2 \int_{\Omega} \left(\frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi \right)^2 G_{\lambda} dx \\
&= \mathcal{B} - 2 \int_{\Omega} \Delta \varphi \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi G_{\lambda} dx - \frac{1}{(L - t + \lambda)} D_{\lambda}(t) \\
&\quad + 2 \int_{\Omega} \left(\frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi \right)^2 G_{\lambda} dx.
\end{aligned}$$

This, together with (3.20), yields that

$$\begin{aligned}
 D'_\lambda(t) &= -\mathcal{B} - 2 \int_{\Omega} \left(\frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi \right)^2 G_\lambda dx + 2 \int_{\Omega} (\Delta \varphi + \partial_t \varphi) \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi G_\lambda dx \\
 &\quad - 2 \int_{\Omega} \partial_t \varphi \Delta \varphi G_\lambda dx + \frac{1}{(L - t + \lambda)} D_\lambda(t) \\
 &= -\mathcal{B} - 2 \int_{\Omega} \left[\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right]^2 G_\lambda dx \\
 &\quad + 2 \int_{\Omega} \frac{1}{4} (\Delta \varphi + \partial_t \varphi)^2 G_\lambda dx - 2 \int_{\Omega} \partial_t \varphi \Delta \varphi G_\lambda dx + \frac{1}{(L - t + \lambda)} D_\lambda(t). \quad (3.23)
 \end{aligned}$$

Here, we have used the following identity:

$$\begin{aligned}
 &\left[\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right]^2 \\
 &= \left(\frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi \right)^2 - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi (\Delta \varphi + \partial_t \varphi) + \frac{1}{4} (\Delta \varphi + \partial_t \varphi)^2.
 \end{aligned}$$

Finally, (3.23) can be rewritten as:

$$\begin{aligned}
 D'_\lambda(t) &= -\mathcal{B} - 2 \int_{\Omega} \left[\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right]^2 G_\lambda dx \\
 &\quad + \frac{1}{2} \int_{\Omega} (\Delta \varphi - \partial_t \varphi)^2 G_\lambda dx + \frac{1}{(L - t + \lambda)} D_\lambda(t). \quad (3.24)
 \end{aligned}$$

Step 5. Computation of the term $N'_\lambda(t)$ over the interval $[0, L]$.

We infer from (3.11), (3.24) and (3.19) that

$$\begin{aligned}
 N'_\lambda(t) &= 2 \left(\frac{1}{H_\lambda(t)} \right)^2 [D'_\lambda(t) H_\lambda(t) - D_\lambda(t) H'_\lambda(t)] \\
 &= \frac{2}{H_\lambda(t)} \left\{ -\mathcal{B} + \frac{1}{(L - t + \lambda)} D_\lambda(t) \right. \\
 &\quad \left. - 2 \int_{\Omega} \left[\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right]^2 G_\lambda dx + \frac{1}{2} \int_{\Omega} (\Delta \varphi - \partial_t \varphi)^2 G_\lambda dx \right\}
 \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{H_\lambda(t)} \right)^2 \left\{ - \left[2 \int_{\Omega} \varphi \left(\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right) G_\lambda dx \right]^2 \right. \\
& \left. + \left[\int_{\Omega} \varphi (\Delta \varphi - \partial_t \varphi) G_\lambda dx \right]^2 \right\}
\end{aligned}$$

from which, we deduce that

$$\begin{aligned}
N'_\lambda(t) &= \frac{1}{L - t + \lambda} N_\lambda(t) - 2 \frac{\mathcal{B}}{H_\lambda(t)} \\
&+ \left(\frac{2}{H_\lambda(t)} \right)^2 \left\{ \left[\int_{\Omega} \varphi \left(\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right) G_\lambda dx \right]^2 \right. \\
&- \int_{\Omega} \left[\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right]^2 G_\lambda dx \int_{\Omega} |\varphi|^2 G_\lambda dx \left. \right\} \\
&- \left(\frac{1}{H_\lambda(t)} \right)^2 \left[\int_{\Omega} \varphi (\Delta \varphi - \partial_t \varphi) G_\lambda dx \right]^2 + \frac{1}{H_\lambda(t)} \int_{\Omega} (\Delta \varphi - \partial_t \varphi)^2 G_\lambda dx. \quad (3.25)
\end{aligned}$$

Step 6. Estimate of the term $\frac{d}{dt}[(L - t + \lambda)N_\lambda(t)]$ over the interval $[0, L]$.

By Cauchy–Schwarz inequality, we have

$$\begin{aligned}
& \left[\int_{\Omega} \varphi \left(\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right) G_\lambda dx \right]^2 \\
& \leq \int_{\Omega} \left[\partial_t \varphi - \frac{x - x_0}{2(L - t + \lambda)} \nabla \varphi + \frac{1}{2} (\Delta \varphi - \partial_t \varphi) \right]^2 G_\lambda dx \int_{\Omega} |\varphi|^2 G_\lambda dx.
\end{aligned}$$

Applying the above inequality to the identity (3.25), we get the estimate:

$$N'_\lambda(t) - \frac{1}{L - t + \lambda} N_\lambda(t) + 2 \frac{\mathcal{B}}{H_\lambda(t)} - \frac{1}{H_\lambda(t)} \int_{\Omega} (\Delta \varphi - \partial_t \varphi)^2 G_\lambda dx \leq 0. \quad (3.26)$$

Now we study the term \mathcal{B} (see (3.22)). We first observe:

- (i) Since $\varphi = 0$ on $\partial\Omega$, it holds that $\nabla \varphi = \partial_\nu \varphi \nu$ on $\partial\Omega$.
- (ii) Because the domain Ω is convex, we necessarily have that $((x - x_0) \cdot \nu) \geq 0$.

Then, the above-mentioned two observations, together with (3.15), show that

$$\begin{aligned}
 \mathcal{B} &= \int_{\partial\Omega} |\nabla\varphi|^2 \partial_\nu G_\lambda \, d\sigma - 2 \int_{\partial\Omega} \partial_\nu \varphi (\nabla\varphi \nabla G_\lambda) \, d\sigma \\
 &= -\frac{1}{2(L-t+\lambda)} \int_{\partial\Omega} |\nabla\varphi|^2 ((x-x_0) \cdot \nu) G_\lambda(x, t) \, d\sigma + \frac{1}{(L-t+\lambda)} \\
 &\quad \times \int_{\partial\Omega} \partial_\nu \varphi ((x-x_0) \nabla\varphi) G_\lambda \, d\sigma \\
 &= -\frac{1}{2(L-t+\lambda)} \int_{\partial\Omega} |\nabla\varphi|^2 ((x-x_0) \cdot \nu) G_\lambda(x, t) \, d\sigma + \frac{1}{(L-t+\lambda)} \\
 &\quad \times \int_{\partial\Omega} |\partial_\nu \varphi|^2 ((x-x_0) \cdot \nu) G_\lambda \, d\sigma \\
 &= \frac{1}{2(L-t+\lambda)} \int_{\partial\Omega} |\partial_\nu \varphi|^2 ((x-x_0) \cdot \nu) G_\lambda(x, t) \, d\sigma \geq 0.
 \end{aligned}$$

This, combined with (3.26), gives the following inequality:

$$(L-t+\lambda)N'_\lambda(t) - N_\lambda(t) - \frac{(L-t+\lambda)}{H_\lambda(t)} \int_{\Omega} (\Delta\varphi - \partial_t\varphi)^2 G_\lambda \, dx \leq 0,$$

from which, we obtain that

$$\frac{d}{dt}[(L-t+\lambda)N_\lambda(t)] \leq (L+\lambda)\|a\|_{L^\infty(\Omega \times (0,L))}^2 \quad \forall t \in [0, L]. \quad (3.27)$$

Step 7. Boundedness of $\lambda N_\lambda(L) + \frac{d}{2}$.

In what follows, we shall derive the boundedness of the term $\lambda N_\lambda(L)$ from (3.27). First of all, we observe that

$$\begin{aligned}
 \lambda N_\lambda(L) &\leq (L-t+\lambda)N_\lambda(t) + (L+\lambda)(L-t)\|a\|_{L^\infty(\Omega \times (0,L))}^2 \\
 &\leq (L+\lambda)N_\lambda(t) + (L+\lambda)L\|a\|_{L^\infty(\Omega \times (0,L))}^2 \quad \forall t \in [0, L].
 \end{aligned}$$

Integrating the above over $(0, L/2)$, we obtain

$$\frac{L}{2}\lambda N_\lambda(L) \leq (L+\lambda) \int_0^{L/2} N_\lambda(t) \, dt + (L+\lambda)\frac{L^2}{2}\|a\|_{L^\infty(\Omega \times (0,L))}^2.$$

On the other hand, by integrating (3.18) over $(0, L/2)$, we get

$$\begin{aligned} \int_0^{L/2} N_\lambda(t) dt &= - \int_0^{L/2} \frac{H'_\lambda(t)}{H_\lambda(t)} dt + \int_0^{L/2} \frac{2}{H_\lambda(t)} \int_\Omega \varphi(\partial_t \varphi - \Delta \varphi) G_\lambda dx dt \\ &\leq \ln \frac{H_\lambda(0)}{H_\lambda(L/2)} + L \|a\|_{L^\infty(\Omega \times (0, L))}. \end{aligned}$$

Now, we derive from the above-mentioned two inequalities that

$$\frac{L}{2} \lambda N_\lambda(L) \leq (L + \lambda) \left[\ln \left(\frac{H_\lambda(0)}{H_\lambda(L/2)} \right) + L \|a\|_{L^\infty(\Omega \times (0, L))} + \frac{L^2}{2} \|a\|_{L^\infty(\Omega \times (0, L))}^2 \right].$$

Since

$$\frac{H_\lambda(0)}{H_\lambda(L/2)} \leq \frac{\int_\Omega |\varphi(x, 0)|^2 dx}{\int_\Omega |\varphi(x, L/2)|^2 dx} \frac{(L/2 + \lambda)^{d/2}}{(L + \lambda)^{d/2}} e^{\frac{m}{4(L/2 + \lambda)}} \leq e^{\frac{m}{2L}} \frac{\int_\Omega |\varphi(x, 0)|^2 dx}{\int_\Omega |\varphi(x, L/2)|^2 dx},$$

it holds that

$$\begin{aligned} \frac{L}{2} \lambda N_\lambda(L) &\leq (L + \lambda) \left[\frac{m}{2L} + \ln \left(\frac{\int_\Omega |\varphi(x, 0)|^2 dx}{\int_\Omega |\varphi(x, L/2)|^2 dx} \right) \right. \\ &\quad \left. + L \|a\|_{L^\infty(\Omega \times (0, L))} + \frac{L^2}{2} \|a\|_{L^\infty(\Omega \times (0, L))}^2 \right]. \end{aligned}$$

Finally, by utilizing the estimate

$$\frac{\int_\Omega |\varphi(x, L)|^2 dx}{\int_\Omega |\varphi(x, L/2)|^2 dx} \leq e^{L \|a\|_{L^\infty(\Omega \times (L/2, L))}},$$

we obtain the desired inequality as the following:

$$\begin{aligned} \lambda N_\lambda(L) &\leq \left(\frac{\lambda}{L} + 1 \right) \left[2 \ln \left(\frac{\int_\Omega |\varphi(x, 0)|^2 dx}{\int_\Omega |\varphi(x, L)|^2 dx} \right) + \frac{m}{L} \right. \\ &\quad \left. + 4L \|a\|_{L^\infty(\Omega \times (0, L))} + L^2 \|a\|_{L^\infty(\Omega \times (0, L))}^2 \right]. \end{aligned}$$

This completes the proof of Lemma 3.1. \square

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